

# Secure and Privacy-Preserving Average Consensus

Minghao Ruan and Yongqiang Wang<sup>1</sup>

**Abstract**—Average consensus is fundamental for distributed systems since it underpins key functionalities of such systems ranging from distributed information fusion, decision-making, to decentralized control. In order to reach an agreement, existing average consensus algorithms require each agent to exchange explicit state information with its neighbors. This leads to the disclosure of private state information, which is undesirable in cases where privacy is of concern. In this paper, we propose a novel approach that enables secure and privacy-preserving average consensus in a decentralized architecture in the absence of any trusted third-parties. By leveraging homomorphic cryptography, our approach can guarantee consensus to the exact value in a deterministic manner. The proposed approach is light-weight in computation and communication, and applicable to time-varying interaction topology cases. Implementation details and numerical examples are provided to demonstrate the capability of our approach.

## I. INTRODUCTION

As a building block of distributed computing, average consensus has been an active research topic in computer science and optimization for decades [1], [2]. In recent years, with the advances of wireless communications and embedded systems, particularly the advent of wireless sensor networks and the Internet-of-Things, average consensus is finding increased applications in fields as diverse as automatic control, signal processing, social sciences, robotics, and optimization [3].

Conventional average consensus approaches employ the explicit exchange of state values among neighboring nodes to reach agreement on the average computation. Such an explicit exchange of state information has two disadvantages. First, it results in breaches of the privacy of participating nodes who want to keep their data confidential. For example, a group of individuals using average consensus to compute a common opinion may want keep secret their individual personal opinions [4]. Another example is power systems where multiple generators want to reach agreement on cost while keeping their individual generation information private [5]. Secondly, storing or exchanging information in the plaintext form (without encryption) is vulnerable to attackers which try to steal information by hacking into the communication links or even the nodes. With the increased number of reported attack events and the growing awareness of security, keeping data encrypted in storage and communications has become the norm in many applications, particularly many real-time sensing and control systems such as the power systems and wireless sensor networks.

To address the pressing need for privacy and security, recently, several relevant average consensus approaches have been proposed. Most of these approaches use the idea of obfuscation to mask the true state values by adding carefully-designed noise on the state. Such approaches usually exploit tools such as mean-square statistics [6] or “differential privacy” which is heavily used for database privacy in computer science [7]–[9]. Although enhances privacy, such noise-based obfuscation also unavoidably affects the performance of average consensus, either directly preventing converging to the true value, or making convergence only achievable in the statistical mean-square sense. Furthermore, these approaches normally rely on the assumption of time-invariant interaction graph, which is difficult to satisfy in many practical applications where the interaction patterns may vary due to node mobility or fading communication channels.

Neither can the above noise-based approaches protect nodes from attackers which try to steal information by hacking into the nodes or the communication channels. To improve resilience to such attacks, a common approach is to employ cryptography. However, it is worth noting that although cryptography based approaches can easily provide privacy and security when a trusted third-party is available, like in the multi-party computation [10], their extension to completely *decentralized* average consensus without any *trusted* third-parties are extremely difficult due to the difficulties in the decentralized management of keys. In fact, in the only reported result incorporating cryptography into decentralized average consensus [10], privacy is obtained by paying the price of depriving participating nodes from access to the final consensus value, although partial information such as a binary decision is still retrievable for participating nodes.

In this paper, we propose a homomorphic cryptography based approach that can guarantee privacy and security in decentralized average consensus even in the presence of a time-varying interaction graph. Different from existing noise-based privacy-preserving approaches which can only achieve average consensus in the statistic case, our approach can guarantee convergence to the *exact* average value in a *deterministic* manner. Unlike the existing cryptography based average consensus approach in [11], this approach allows every participating nodes to access the exact final value. Furthermore, the approach is completely decentralized and light-weight in computation, which makes it easily applicable to resource restricted systems. Simulations results are given to verify the approach.

The outline of this paper is as follows. Sections II reviews the protocol used for average consensus problems and the

<sup>1</sup>Minghao Ruan and Yongqiang Wang are with the Department of Electrical Engineering, Clemson University, Clemson, SC 29634, USA mruan, yongqiw@clemson.edu

homomorphic cryptography, particularly the Paillier cryptosystem. Our encrypted protocol is introduced in Section III. In Section IV we provide a proof of convergence as well as bounds of critical parameter. The extension to weighted average consensus is also presented. Some implementation issues are discussed in Section V and numerical examples are presented in Section VI, followed by a discussion of security issues in Section VII.

## II. BACKGROUND

In this section we will briefly review the average consensus problem and the homomorphic cryptography.

### A. Average Consensus

We follow the same convention used in [3] where a network of  $n$  agents is represented by a graph  $G = (V, E, A)$  with nodes  $V = \{v_1, v_2, \dots, v_n\}$ , the set of edges  $E \subset V \times V$ , and a weighted adjacency matrix  $A = [a_{ij}]$  which satisfies  $a_{ij} > 0$  if  $(v_i, v_j) \in E$  and 0 otherwise. The set of neighbors of a node  $v_i$  is defined as

$$N_i = \{v_j \in V | (i, j) \in E\} \quad (1)$$

We assume the graph is undirected so that  $A$  is symmetric

$$a_{ij}^{(t)} = a_{ji}^{(t)} > 0 \quad \forall (i, j) \in E \quad (2)$$

Note that we use the superscript  $t$  to denote time-varying weights. To achieve the average consensus, namely converging to the average of all initial values, one commonly used protocol for continuous time (CT) is

$$\dot{x}_i(t) = \sum_{j \in N_i} a_{ij}^{(t)} \cdot (x_j(t) - x_i(t)) \quad (3)$$

and the counterpart for discrete time (DT) is:

$$x_i[k+1] = x_i[k] + \varepsilon \sum_{j \in N_i} a_{ij}^{(k)} \cdot (x_j[k] - x_i[k]) \quad (4)$$

where  $\varepsilon$  is a constant step size in the range  $(0, 1]$ .

### B. Homomorphic Encryption

Our method to protect privacy and security is by encrypting the states. Most popular encryption algorithms such as RSA [12], ElGamal [13] and Paillier [14] are public-key cryptographies in which anyone with the public key can encrypt a message but the encrypted message can only be decrypted by the private key. In this paper we focus on the Paillier cryptosystem which provides the following basic functions:

- Key generation:
  - 1) Choose two large prime numbers  $p$  and  $q$  of equal bit-length and compute  $n = pq$ .
  - 2) Let  $g = n + 1$ .
  - 3) Let  $\lambda = \phi(n) = (p-1)(q-1)$  where  $\phi(\cdot)$  is the Euler's totient function.
  - 4) Let  $\mu = \phi(n)^{-1} \bmod n$  which is the modular multiplicative inverse of  $\phi(n)$ .
  - 5) The public key  $k_p$  is then  $(n, g)$ .
  - 6) The private key  $k_s$  is then  $(\lambda, \mu)$ .

- Encryption ( $c = \mathcal{E}(m)$ ):
  - 1) Choose a random  $r \in \mathbb{Z}_n^*$ .
  - 2) The ciphertext is given by  $c = g^m \cdot r^n \bmod n^2$ , where  $m \in \mathbb{Z}_n, c \in \mathbb{Z}_{n^2}^*$ .
- Decryption ( $m = \mathcal{D}(c)$ ):
  - 1) Define the integer division function  $L(u) = \frac{u-1}{n}$ .
  - 2) The plaintext is  $m = L(c^\lambda \bmod n^2) \cdot \mu \bmod n$ .

An encryption algorithm is homomorphic if it allows certain computations to be carried out on the encrypted ciphertext. The Paillier cryptosystem is shown to be additive homomorphic:

$$\begin{aligned} \mathcal{E}(m_1, r_1) \cdot \mathcal{E}(m_2, r_2) &= (g_1^{m_1} r_1^n) \cdot (g_1^{m_2} r_2^n) \bmod n^2 \\ &= (g_1^{m_1+m_2} (r_1 r_2)^n) \bmod n^2 \\ &= \mathcal{E}(m_1 + m_2, r_1 r_2) \end{aligned} \quad (5)$$

The dependency on random numbers  $r_1$  and  $r_2$  are explicitly shown in (5), yet they play no role in the decryption. The following shorthand notation is often used instead:

$$\mathcal{E}(m_1) \cdot \mathcal{E}(m_2) = \mathcal{E}(m_1 + m_2) \quad (6)$$

Moreover, if we multiply the same ciphertext  $k \in \mathbb{Z}^+$  times

$$\mathcal{E}(m)^k = \prod_{i=1}^k \mathcal{E}(m) = \mathcal{E}\left(\sum_{i=1}^k m\right) = \mathcal{E}(km) \quad (7)$$

Notice however, Paillier is not multiplicative homomorphic because  $k$  in (7) is in plaintext. Furthermore, the existence of the random number  $r$  in Paillier cryptosystem gives it resistance to dictionary attacks which infer a key to an encrypted message by systematically trying every possibilities, such as words in a dictionary.

## III. THE ENCRYPTED PROTOCOL

The main contribution of this paper is a completely decentralized, trusted-third-party free protocol that guarantees average consensus while protecting the privacy of the participants. Instead of adding noise to the states cover the states information, our approach combines encryption and randomness in the system dynamics, i.e. the coupling weights  $a_{ij}^{(t)}$ . This way the states are free from being contaminated by covering noise, and thus guarantees a deterministic convergence to the exact average value.

In this section we present details of our encrypted protocol based on (3) and (4). In particular we focus on the weighted difference between two connected nodes:

$$\begin{aligned} \Delta x_{ij} &= a_{ij}^{(t)} \cdot (x_j - x_i) \\ \Delta x_{ji} &= a_{ji}^{(t)} \cdot (x_i - x_j) \end{aligned} \quad (8)$$

subject to  $a_{ij}^{(t)} = a_{ji}^{(t)} > 0$

We call each time (8) is computed between a pair of nodes an *exchange* of states. For continuous time system (3) becomes

$$\dot{x}_i(t) = \sum_{j \in N_i} \Delta x_{ij}(t) \quad (9)$$

For discrete time let  $\Delta x_{ij}[k] = \Delta x_{ij}[k]$ , (4) becomes

$$x_i[k+1] = x_i[k] + \varepsilon \sum_{j \in N_i} \Delta x_{ij}[k] \quad (10)$$

Notice that in a decentralized system it is not possible to protect the privacy of both nodes if the protocol (8) is directly used. This is due to the fact that even if we encrypt all the intermediate steps, if one node, for instance  $i$ , has access to  $a_{ij}$ , it can still infer the value of  $x_j$  (i.e.  $x_j = \frac{\Delta x_{ij}}{a_{ij}} + x_i$ ).

We solve this problem by decoupling each weight  $a_{ij}$  as the product of two random numbers, namely  $a_{ij} = a_i \cdot a_j = a_{ji}$ , with  $a_i > 0$  only known to node  $v_i$  and  $a_j > 0$  only known to node  $v_j$ . We will show later that this decoupled weight approach renders two interacting nodes unable to infer the other node's state while guaranteeing convergence to the average. Next, without loss of generality, we consider a pair of connected nodes ( $v_1, v_2$ ) to illustrate the idea (cf. Fig. 1). For simplicity, we assume the states  $x_1$  and  $x_2$  are scalar. Each node maintains its own public and private key pairs  $(k_p, k_s)_i$ ,  $i = \{1, 2\}$ .

Due to symmetry, we only show how node  $v_1$  obtains the weighted state difference, i.e. the flow  $v_1 \rightarrow v_2 \rightarrow v_1$ . Before the start of the information exchange, node  $v_1$  (resp.  $v_2$ ) generates its new positive random number  $a_1$  (resp.  $a_2$ ). First, node  $v_1$  sends its encrypted negative state  $\mathcal{E}_1(-x_1)$  as well as the public key  $k_{p1}$  to node  $v_2$ . Node  $v_2$  then computes the encrypted  $a_2$ -weighted difference  $\mathcal{E}_1(a_2(x_2 - x_1))$  as follows:

- 1) Encrypt  $x_2$  with  $v_1$ 's public key  $k_{p1}$ :  $x_2 \rightarrow \mathcal{E}_1(x_2)$ .
- 2) Compute the difference directly in ciphertext:

$$\mathcal{E}_1(x_2 - x_1) = \mathcal{E}_1(x_2 + (-x_1)) = \mathcal{E}_1(x_2) \cdot \mathcal{E}_1(-x_1) \quad (11)$$

- 3) Compute the  $a_2$ -weighted difference in ciphertext:

$$\mathcal{E}_1(a_2 \cdot (x_2 - x_1)) = (\mathcal{E}_1(x_2 - x_1))^{a_2} \quad (12)$$

- 4) The result from (12) is sent back to  $v_1$ .

Lastly,  $v_1$  decrypts the message with the private key  $k_{s1}$  and multiplies the result with  $a_1$  to get the (balanced-)weighted difference:

$$\begin{aligned} \mathcal{E}_1(a_2(x_2 - x_1)) &\xrightarrow{\mathcal{D}_1} a_2(x_2 - x_1) \\ \Delta x_{12} &= a_1 a_2(x_2 - x_1) \end{aligned} \quad (13)$$

In a similar manner, the exchange  $v_2 \rightarrow v_1 \rightarrow v_2$  produces  $\mathcal{E}_2(a_1(x_1 - x_2))$  for  $v_2$  who then decrypts the message and multiplies the result by its own multiplier  $a_2$  to get  $\Delta x_{21}$

$$\begin{aligned} \mathcal{E}_2(a_1(x_1 - x_2)) &\xrightarrow{\mathcal{D}_2} a_1(x_1 - x_2) \\ \Delta x_{21} &= a_2 a_1(x_1 - x_2) \end{aligned} \quad (14)$$

After each node collects the weighted differences from all neighbors, it updates the state with (3) or (4) accordingly.

It is worth noting that:

- $v_2$  cannot see  $x_1$  which is encrypted in  $\mathcal{E}_1(-x_1)$ .
- Given  $a_2(x_2 - x_1)$ ,  $v_1$  cannot solve for  $x_2$  because  $a_2$  is only known to  $v_2$ .

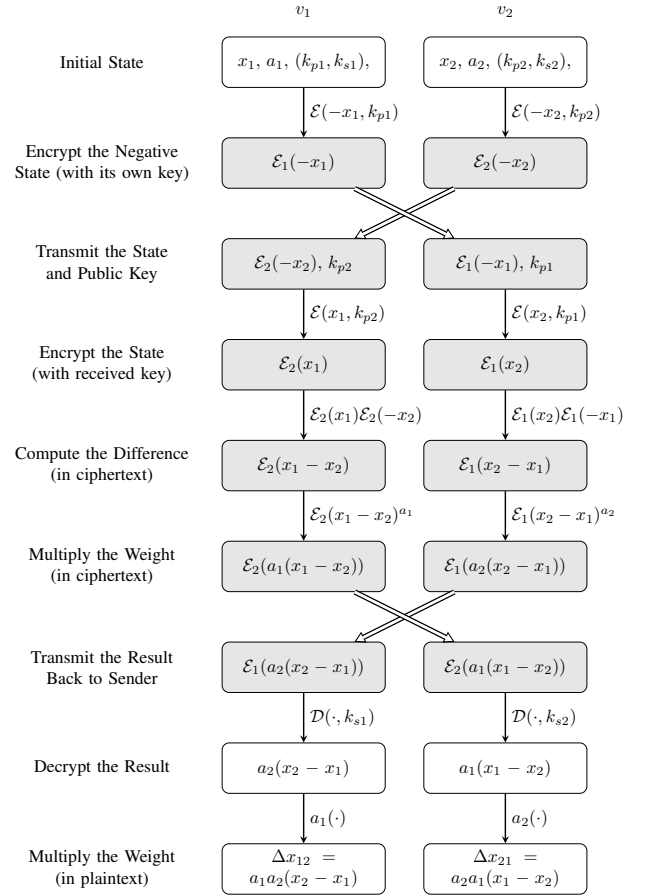


Fig. 1: A step-by-step illustration of the exchange protocol. Single arrows indicates the flow of steps; double arrows indicate data exchange via a communication channel. Shaded nodes indicate the computation done in ciphertext.

- The only time both nodes know each other's state is when  $x_1 = x_2$ , in which case  $\Delta x_{12} = \Delta x_{21} = 0$  regardless of the choice of  $a_1$  and  $a_2$ .
- We encrypt  $\mathcal{E}_1(-x_1)$  because it is much more difficult to compute subtraction in ciphertext. The issue regarding encrypting negative values using Paillier is discussed in section V.

#### IV. THEORETICAL ANALYSIS

It has been proved in [3] that given the protocol in (3) and (4), the system will converge to the average. To keep this paper self-contained, we provide a brief proof which also helps shed some light on how to set the appropriate values for the multiplier  $a_i$  (and  $\varepsilon$  for DT). The extension to weighted average consensus is also presented.

##### A. Convergence for Continuous Time System

Let  $x \in \mathbb{R}^n$  denote the graph state. The node dynamics in (3) can be written as:

$$\dot{x} = -L^{(t)}x(t) \quad (15)$$

where  $L^{(t)} = [l_{ij}^{(t)}]$  is the time-varying Laplacian matrix defined by

$$l_{ij}^{(t)} = \begin{cases} \sum_{j \in N_i} a_{ij}^{(t)} & i = j \\ -a_{ij}^{(t)} & i \neq j \end{cases} \quad (16)$$

**Theorem 1:** Given  $L^{(t)}$  defined in (16) with  $a_{ij} > 0$ , the system converges to the average given by

$$\lim_{t \rightarrow \infty} x(t) = \alpha \mathbf{1} \quad \text{with } \alpha = \text{Avg}(0) \quad (17)$$

*Proof:* From the definition of  $L^{(t)}$  and the discussion in Section III we know that  $L^{(t)}$  is symmetric and the row/column sums of  $L^{(t)}$  are 0. This means that  $\mathbf{1}^T/\mathbf{1}$  is a left/right eigenvector of  $L^{(t)}$  with eigenvalue 0.

For continuous time system the average consensus is time-invariant, i.e.,  $\text{Avg}(t) = \text{Avg}(0)$  because

$$\begin{aligned} \frac{d}{dt} \text{Avg}(t) &= \frac{d}{dt} \left( \frac{1}{n} \mathbf{1}^T x(t) \right) \\ &= -\frac{1}{n} \mathbf{1}^T L^{(t)} x(t) = 0 \end{aligned} \quad (18)$$

Therefore  $\text{Avg}(x(t)) = \text{Avg}(x(0)) = \alpha$ , and we can decompose the state  $x(t)$  as a sum of the constant steady state  $\alpha \mathbf{1}$  plus the *disagreement* vector  $\delta(t)$  orthogonal to  $\mathbf{1}$ :

$$x(t) = \alpha \mathbf{1} \oplus \delta(t) \quad (19)$$

It follows directly that  $\delta(t)$  evolves with the same dynamics:

$$\dot{\delta}(t) = -L^{(t)} \delta(t) \quad (20)$$

We can define the Lyapunov function  $\Phi(\delta)$  and its derivative:

$$\Phi(\delta) = \frac{1}{2} \|\delta\|^2 \geq 0 \quad (21)$$

$$\dot{\Phi}(\delta) = \delta^T \dot{\delta} = -\delta^T L^{(t)} \delta \quad (22)$$

To prove that the system converges to the average consensus, it suffices to show  $\delta \rightarrow 0$  as  $t \rightarrow \infty$ . By Gershgorin's circle theorem each eigenvalue of  $L$  lies within at least one of the disks:

$$\exists i : \quad \|\lambda - l_{ii}^{(t)}\| \leq \sum_{j \neq i} |l_{ij}^{(t)}| \quad (23)$$

Since  $l_{ii}^{(t)} = \sum_{j \neq i} |l_{ij}^{(t)}|$ , all the disks touch the origin on the left. Given  $L$  is real and symmetric, which means all the other eigenvalues are real and positive. Let  $\Delta^{(t)} = \max_i l_{ii}^{(t)}$ , all  $\lambda$ s satisfy

$$0 = \lambda_1 < \lambda_2 \leq \dots \leq \lambda_n \leq 2\Delta^{(t)} \quad (24)$$

Furthermore, since  $\delta \neq 0$  and  $\delta$  orthogonal to  $\mathbf{1}$ ,

$$\min_{\delta \neq 0, \delta \perp \mathbf{1}} \frac{\delta^T L^{(t)} \delta}{\delta^T \delta} = \lambda_2 > 0 \quad (25)$$

Therefore  $\dot{\Phi}(\delta)$  is always negative

$$\dot{\Phi}(\delta) = -\delta^T L^{(t)} \delta \leq -\lambda_2 \delta^T \delta < 0, \quad \forall \delta \neq 0 \quad (26)$$

This result implies that we only have to make sure  $a_i(t) > 0$  to ensure all eigenvalues of  $L$  except the trivial one are positive. The convergence speed is dominated by the second smallest eigenvalue  $\lambda_2$ .  $\square$

## B. Convergence for Discrete Time System

In discrete-time (4) can be written as

$$x[k+1] = P^{(t)} x[k] \quad (27)$$

where  $P^{(t)} = I - \varepsilon L^{(t)}$  is the Perron matrix.

**Theorem 2:** Given  $P^{(t)}$  with  $a_{ij} > 0$  and  $\varepsilon < \frac{1}{\Delta^{(t)}}$ , the system converges to the average given by

$$\lim_{k \rightarrow \infty} x[k] = \alpha \mathbf{1} \quad \text{with } \alpha = \text{Avg}[0] \quad (28)$$

*Proof:* First it is easy to show that the average is invariant to time step  $k$ , i.e.,  $\text{Avg}[k] = \text{Avg}[0] = \alpha$ :

$$\begin{aligned} \text{Avg}[k+1] - \text{Avg}[k] &= \frac{\mathbf{1}^T}{n} x[k+1] - \frac{\mathbf{1}^T}{n} x[k] \\ &= \frac{\mathbf{1}^T}{n} (-\varepsilon L^{(t)}) x[k] = 0 \end{aligned} \quad (29)$$

Let  $x[k] = \alpha \mathbf{1} \oplus \delta[k]$ , where  $\delta[k]$  is orthogonal to  $\mathbf{1}$ . We want to show  $\delta[k] \rightarrow 0$  as  $k \rightarrow \infty$ .  $P^{(t)}$ 's eigenvalue  $\mu_i$  is related to  $\lambda_i$  by

$$\begin{aligned} \mu_i &= 1 - \varepsilon \lambda_i \\ 1 - 2\varepsilon \Delta^{(t)} \leq \mu_n \leq \dots \leq \mu_2 < \mu_1 &= 1 \end{aligned} \quad (30)$$

Since

$$\delta[k+1] = P^{(t)} \delta[k] \quad (31)$$

For  $x[k]$  to converge to  $\alpha \mathbf{1}$ , the norm of  $\delta[k]$  must monotonically decrease, i.e.

$$\begin{aligned} \|\delta[k+1]\| &= \|P^{(t)} \delta[k]\| \\ &\leq \|P^{(t)}\| \cdot \|\delta[k]\| \\ &\leq \max_i |\mu_i| \cdot \|\delta[k]\| \\ &< \|\delta[k]\| \end{aligned} \quad (32)$$

The last inequality requires all the eigenvalues of  $P^{(t)}$  lie within the unit circle. A sufficient condition for this to happen is when

$$-1 < 1 - 2\varepsilon \Delta^{(t)} \Rightarrow \varepsilon < 1/\Delta^{(t)} \quad (33)$$

$\square$

Although  $\Delta$  is time varying and difficult to compute, we can always set the individual weight  $0 < a_i < 1$  so that

$$\Delta = \max_{i \in V} \left( \sum_{v_j \in N_i} a_i a_j \right) < \max_{v_i \in V} |N_i| \quad (34)$$

where  $|N_i|$  is the number of neighbors. To ensure convergence, it is convenient to set

$$\varepsilon = \frac{1}{\max_i |N_i|} < \frac{1}{\Delta^{(t)}} \quad (35)$$

*Remark 1:* Since the framework allows time-varying weighted matrix  $A^{(t)}$ , it can easily be extended to the case with time-varying interaction graphs according to [12] and [15].

### C. Weighted Average Consensus

Using the same encrypted protocol, the weighted average consensus can be achieved easily by introducing a fixed weight  $w_i$  with  $w_i > 0$ . Without loss of generality, we assume  $\sum_i^n w_i = 1$ .

For continuous time (9) becomes

$$\dot{x}_i(t) = \frac{1}{w_i} \sum_{j \in N_i} \Delta x_{ij}(t) \quad (36)$$

For discrete time let (10) becomes

$$x_i[k+1] = x_i[k] + \frac{\varepsilon}{w_i} \sum_{j \in N_i} \Delta x_{ij}[k] \quad (37)$$

**Theorem 3:** Given the node dynamics in (36) and (37), the network will converge to a weighted average consensus. Concretely,

$$\lim_{t \rightarrow \infty} x(t) = \lim_{k \rightarrow \infty} x[k] = \alpha \mathbf{w}^T \quad (38)$$

where  $\alpha = \text{Avg}(0)$ . The proof is similar to that of the average consensus. A complete proof can be found in [16]. To guarantee convergence in discrete time,  $\varepsilon$  must now satisfy  $\varepsilon < \frac{\min w_i}{\Delta(t)}$ .

## V. IMPLEMENTATION DETAILS

In addition to the constraints imposed on  $a_i$  and  $\varepsilon$ . There are other technical issues that must be addressed to actually implement the protocol.

### A. Quantization

Real-world applications typically have  $x_i \in \mathbb{R}$  which are represented by floating point numbers in modern computing architectures. On the contrary, encryption algorithms only work on unsigned integers. In our case, we are interested in converting  $x_i$  to fixed length integers.

We convert the state  $x_i$  in floating point representation to a (signed) integer by uniformly multiply a sufficiently large number  $N_{max}$  to all  $x_i$ s and round off the fraction. At the end of the computation,  $N_{max}$  is divided from the result to recover the real value.

### B. Subtraction and Negative Values

Another issue is how to treat the sign of an integer for encryption. [17] solves this problem by mapping negative values to the end of the group  $\mathbb{Z}_n$  where  $n = pq$  is given by the public key. We offer an alternative solution by taking advantages of the fact that encryption algorithms blindly treat bit strings as an unsigned integer. In our implementation all integer values are stored in fix-length integers (i.e. long int in C) and negative values are left in 2's complement format. Encryption and intermediate computations are carried out as if the underlying data were unsigned. When the final message is decrypted, the overflown bits (bits outside the fixed length) are discarded and the remaining binary number is treated as a signed integer which is later converted back to a real value.

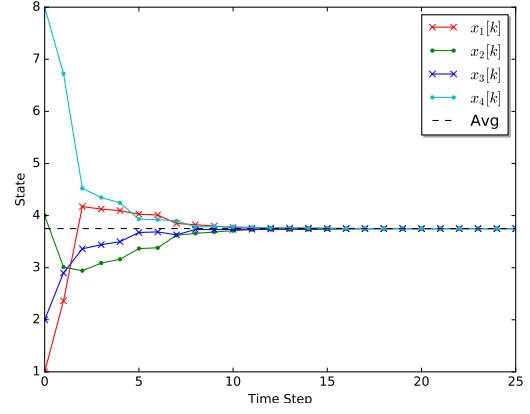


Fig. 2: Convergence to the average consensus. The states converge to the average consensus 3.75 in about 10 steps.

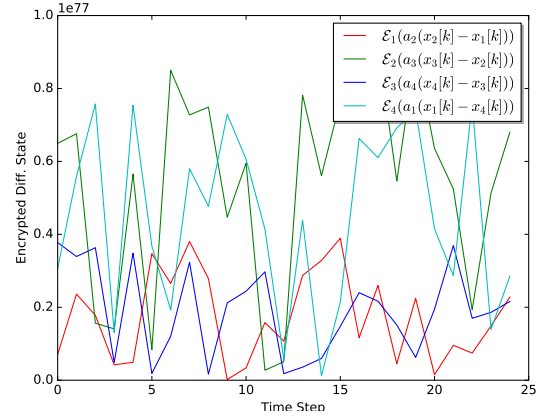


Fig. 3: Encrypted weighted difference vs time step. Although the states have converged after 10 steps, the encrypted difference still appear to be random.

### C. Representation of $a_i$

Since the randomness in  $a_i$  is more important than the actual value of  $a_i$  as long as they satisfy the constraints, it suffices to leave  $a_i$  as positive integers for continuous time; for discrete time we also scale  $a_i \in \mathbb{R}$  by  $N_a$  for intermediate steps and divide the final result by  $N_a$  at the end.

## VI. NUMERICAL EXAMPLE

To illustrate the capability of our protocol, we implemented the discrete-time version in C/C++. We used an open-source C implementation of the Paillier cryptosystem [18] because it allowed byte-level access. For each exchange between two nodes, the states were converted to 64-bit integers by multiplying  $N_{max} = 10^5$ . The weights  $a_i(t) \in (0, 1)$  were also scaled up by  $N_a = 10^2$  and represented by 64-bit random integers. The encryption/decryption keys were set to 256-bit long.

The first simulation had four nodes and each node was connected to two neighbors, so the step size  $\varepsilon$  was set to 0.5. The initial states were set to  $\{1.0, 2.0, 4.0, 8.0\}$  respectively and the average is 3.75. Each node used a static key pair which was initialized once at the beginning. The states' convergence to the average is shown in Figure 2.

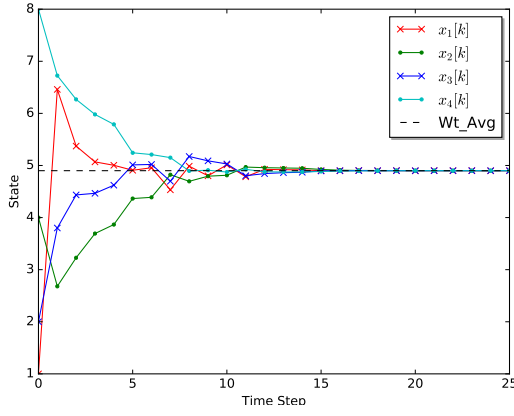


Fig. 4: Convergence to the weighted average consensus. The states converge to the weighted average consensus 4.9 in about 10 steps.

The plot of received messages which encoded the weighted difference between two nodes is shown in Figure 3. It is worth noting that although the states have converged to the average, the encrypted weighted differences still appear to be random to an unintended observer.

The second simulation shows an example of the weighted average consensus. Given the same initial condition as the first example, the nodes also had the associated weights  $\{0.1, 0.2, 0.3, 0.4\}$ . The step size  $\varepsilon$  was set to 0.05. Figure 4 shows the states converge to the weighted average 4.9.

The computational overhead caused by the encryption is manageable. Without any hardware-specific optimization, it takes about 7 ms to compute one exchange of state on a desktop with a 3.4 GHz CPU.

## VII. REMARK ON SECURITY

Our protocol prevents a passive adversary, as a node in the network or as an observer eavesdropping the communication, from stealing the exact states of other nodes in the network. Due to the additive homomorphic property, however, the Paillier cryptosystem is vulnerable to an active adversary who is able to alter the message being sent through the channel. Although this adversary may not find out the exact states of the communicating nodes, she/he can still inflict significant damage to the system.

Consider the scenario where the communication from node  $v_1$  to  $v_2$  is intercepted by a hacker. Since  $v_1$ 's public key  $k_{p1}$  is sent along with  $\mathcal{E}(-x_1)$ , the hacker may use the additive homomorphism to inject an arbitrary noise to modify the original message to  $\mathcal{E}(-x_1 + \xi)$ . Since  $v_2$  has no way to tell if the received message has been modified, the hacker may exploit this vulnerability to make the network either converge to the wrong average consensus or not converge at all.

In applications where security is the primary concern, it is imperative to be able to check the integrity of any incoming message. For this reason it is common to attach a digital signature in addition to the original message (both  $(\mathcal{E}(x_i, k_{pi}))$  and  $\mathcal{E}(a_j(x_j - x_i))$ ) so that the receiving end can verify that the encrypted information has not been altered during the transmission. This can be easily incorporated

into our protocol with a hashing function such as MD5 and another strong encryption algorithm such as RSA with Optimal Asymmetric Encryption Padding (OAEP).

## VIII. CONCLUSIONS

In this paper we proposed a decentralized secure and privacy-preserving protocol for the network average consensus problem. In contrast to previous approaches where the states are covered with random noise which unavoidably affects the convergence performance, we encode randomness to the system dynamics which allows the convergence to the exact average in a deterministic manner. The protocol also provides resilience to passive attackers and allows easy incorporation of active attacker defending mechanisms.

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